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Alternating cycles and trails in 2-edge-coloured complete multigraphs

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Abstract

We consider edge-coloured multigraphs. A trail in such a multigraph is alternating if its successive edges differ in colour. Let G be a 2-edge-coloured complete graph and let M be a 2-edge-coloured complete multigraph. Bankfalvi and Bankfalvi (1968) obtained a necessary and sufficient condition for G to have a Hamiltonian alternating cycle. Generalizing this theorem, Das and Rao (1983) characterized those G which contain a closed alternating trail visiting each vertex v in G exactly $f(v) > 0$ times. We solve the more general problem of determining the length of a longest closed alternating trail T_f visiting each vertex v in M at most $f(v) > 0$ times. Our result is a generalization of a theorem by Saad (1996) that determines the length of a longest alternating cycle in G. We prove the existence of a polynomial algorithm for finding the desired trail T_f. In particular, this provides a solution to a question in Saad (1996).

Keywords: Edge-coloured graphs; Alternating cycles; Alternating trails

1. Introduction

A trail in an edge-coloured multigraph is called *alternating* if its successive edges differ in colour.

In applications to genetics (cf. [8,9]) researchers consider 2-edge-coloured multigraphs which are unions of two monochromatic graphs. From a theoretical point of view there is no good reason to restrict investigation to multigraphs without parallel edges when more general results may be available. Therefore, we shall often deal with 2-edge-coloured multigraphs rather than 2-edge-coloured graphs.

Let G be a 2-edge-coloured complete graph and let M be a 2-edge-coloured complete multigraph. In 1968, solving a problem by Erdős, M. Bankfalvi and Zs. Bankfalvi [2]

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obtained a necessary and sufficient condition for G to have a Hamiltonian alternating cycle. Generalizing this theorem, Das and Rao [7] characterized those G which contain a closed alternating trail visiting each vertex v in G exactly $f(v) > 0$ times. We solve the more general problem of determining the length of a longest closed alternating trail T_f visiting each vertex v in M at most $f(v) > 0$ times. Our result is a generalization of a theorem by Saad [18] that determines the length of a longest alternating cycle in G. We prove the existence of a polynomial algorithm for finding the desired trail T_f. In particular, this provides a solution to the following problem in [18]: is there a polynomial (deterministic) algorithm for finding a longest alternating cycle in a 2-edge-coloured complete graph. The best previous known result is due to Saad [18] who proved the existence of a polynomial random algorithm.

In our proof of the main theorem (Theorem 3.5) we use a very useful connection between directed cycles in bipartite digraphs and alternating cycles in 2-edge-coloured bipartite graphs first discovered by Das [6] (see also [15]). This connection as well as several other relations between directed and alternating trails are also discussed in the survey paper [1].

2. Notation and terminology

The terminology is fairly standard, generally following [3,5]. All graphs, multigraphs and digraphs considered are finite and have no loops. When multigraphs have no parallel edges, we call them *graphs*, as usual. In this paper we deal with *2-edge-coloured multigraphs*, i.e. multigraphs so that each edge has colour 1 (red) or 2 (blue) and no two parallel (i.e. joining the same pair of vertices) edges have the same colour. In the rest of this section, G will stand for a 2-edge-coloured multigraph.

The *red subgraph* (*blue subgraph*, resp.) of G contains the vertices of G and all red (blue, resp.) edges of G. A *trail* is a walk with distinct edges. A walk with distinct vertices is a *path*. A closed trail whose origin and internal vertices are distinct is a *cycle*. In particular, a pair of parallel edges forms a cycle (of length two). A cycle, path or trail in G is called *alternating* if its successive edges differ in colour. In particular, every cycle of length two in a 2-edge-coloured multigraph is alternating (as parallel edges differ in colour). A *cycle subgraph* F of G is a union of alternating cycles in G, all vertex disjoint. A cycle subgraph F of G is *maximum* if F has maximum number of vertices among all cycle subgraphs of G. An alternating path P is called an *(x, y)-path* if x and y are the end vertices of P. An alternating cycle in G is *Hamiltonian* if it contains all vertices of G. A multigraph G is *Hamiltonian* if it has a Hamiltonian alternating cycle.

The colour of an edge e in G will be denoted by $\chi_G(e)$. Let X and Y be two sets of the vertices of G. Then XY denotes the set of all edges having one end vertex in X and the other in Y. In case all the edges in XY have the same colour, say i, we write $\chi_G(XY) = i$. Note that we use the notation $\chi_G(XY)$ only if all the edges in XY have the same colour.

In order to keep notation on multigraphs as simple as possible, we shall sometimes denote an edge with end vertices x and y by xy, even if there are two such edges. In such cases, the complete identification will follow from the context. We shall use this convention not only for distinct edges but also for cycles, paths, etc.

The following notion of colour connectivity was invented by Saad [18] (he used another name for this notion). A pair of vertices x, y of G is called *colour-connected* if there exist alternating (x, y)-paths $P = xx' \dots y'y$ and $P' = xu' \dots v'y$ such that $\chi(xx') \neq \chi(xu')$ and $\chi(y'y) \neq \chi(v'y)$. (Notice that P and P' are paths, not trails.) We define a vertex x to be colour-connected to itself. We say that G is *colour-connected* if every pair of vertices of G is colour-connected.

Clearly, every alternating cycle is a colour-connected graph. This indicates that colour connectivity may be useful for solving alternating cycle problems. We can use colour connectivity more effectively if we know that this is an equivalence relation on the vertices of G. This leads us to the following definition: a multigraph G is *convenient* if colour connectivity is an equivalence relation on the vertices of G. If G is convenient, an equivalence class of colour connectivity is called a *colour-connected component* of G.

Unfortunately, there are non-convenient multigraphs. Consider the graph H on five vertices, $1,2,3,4,5$, and 6 edges, $13,23,45$ of colour 1 and $12,34,35$ of colour 2. It is easy to check that the vertices 1 and 2 are colour-connected to 4, but 1 and 2 are not colour-connected in H.

A multigraph G is *complete* if every two distinct vertices in G are adjacent. Let H be a 2-edge-coloured multigraph with vertices $v_1, \dots, v_k$ $(k \geqslant 2)$. A multigraph L is called an *extension* of H if the vertex set of L can be partitioned into non-trivial subsets $V_1, \dots, V_k$ (called *partite sets*) so that, for every pair i,j $(1 \leqslant i < j \leqslant k)$ and every pair $x \in V_i$, $y \in V_j$, the number of edges between x and y coincides with the number of edges between v_i and v_j (in H; there can be none, one or two edges between v_i and v_j), and if H has only one edge between v_i and v_j, then $\chi_L(V_iV_j) = \chi_H(v_iv_j)$. An *extended 2-edge-coloured complete multigraph* is an extension of a 2-edge-coloured complete multigraph. We denote the set of all extended 2-edge-coloured complete multigraphs by $\mathscr{E}\mathscr{C}\mathscr{M}$.

3. Statements of main results

Let f be a mapping from the vertex set of a 2-edge-coloured complete multigraph G into the set of all *positive* integers. A subgraph H of G is called an *$f_{\geqslant}$-subgraph* of G if $d_{1,H}(x) = d_{2,H}(x) \leqslant f(x)$ for every vertex x in G, where $d_{i,H}(x)$ is the number of edges of colour i in H incident with x. A connected $f_{\geqslant}$-subgraph H of G is called *maximum* if H has maximum number of edges among all connected $f_{\geqslant}$-subgraphs of G. Clearly, if $f(x) = 1$ for every $x \in V(G)$, then the problem of finding a maximum connected $f_{\geqslant}$-subgraph of G coincides with the longest alternating cycle problem. By Kotzig's characterization of edge-coloured graphs which contain connected alternating

Eulerian trails (cf. [11,16]), every connected $f_{\geqslant}$-subgraph of G can be viewed as a closed alternating trail in G visiting each vertex x in G at most $f(x)$ times and vice versa.

In this paper we consider the following problem:

Problem 3.1. Find a maximum connected $f_{\geqslant}$-subgraph in a 2-edge-coloured complete multigraph G on n vertices.

Here we may assume that each $f(x)\leqslant|E(G)|/2$, since every value of $f(x)>|E(G)|/2$ can be replaced by $|E(G)|/2$ without changing the solution of the problem.

It is easy to check that this problem is equivalent to the problem of finding a longest alternating cycle in the extension of G with n partite sets $\{V_x\colon x \in V(G)\}$ of sizes $|V_x| = f(x)$. Therefore, a solution of Problem 3.1 can be obtained from a solution of the following problem:

Problem 3.2. Find a longest alternating cycle in an extended 2-edge-coloured complete multigraph.

Since each $f(x)\leqslant|E(G)|/2$, a polynomial algorithm for solving Problem 3.2 can be converted into a polynomial algorithm for Problem 3.1.

It is easier for us to deal with Problem 3.2 than Problem 3.1. Hence, in the sequel, we shall consider only Problem 3.2.

Obviously, each alternating cycle of a convenient 2-edge-coloured multigraph G is contained in a colour-connected component of G. Hence, we may restrict our attention only to colour-connected multigraphs $G \in \mathscr{E}\mathscr{C}\mathscr{M}$ because of the following two theorems proved in Section 5.

Theorem 3.3. *Every multigraph $G \in \mathscr{E}\mathscr{C}\mathscr{M}$ is convenient.*

Theorem 3.4. *Let G be a convenient 2-edge-coloured multigraph $G = (V,E)$. Then we can check whether G is colour-connected in time $O(|V||E|)$ and find the colour-connected components of G in time $O(|V|^2|E|)$.*

The following result, which is the main result of this paper, generalizes a characterization of longest alternating cycles in 2-edge-coloured graphs [18] (see Corollary 3.6). We prove Theorem 3.5 in Section 4.

Theorem 3.5. *The length of a longest alternating cycle in a colour-connected extended 2-edge-coloured complete multigraph G is equal to the number of vertices in a maximum cycle subgraph of G. Given a maximum cycle subgraph of a colour-connected $G \in \mathscr{E}\mathscr{C}\mathscr{M}$, a longest alternating cycle in G can be constructed in time $O(n^3)$, where n is the number of vertices in G.*

Corollary 3.6 (Saad [18]). *The length of a longest alternating cycle in a colour-connected 2-edge-coloured complete graph H is equal to the number of vertices in a maximum cycle subgraph of H.*

The following theorem is proved in Section 5.

Theorem 3.7. *One can construct a maximum cycle subgraph in a 2-edge-coloured multigraph G on n vertices in time $O(n^3)$.*

Theorems 3.3–3.5 and 3.7 imply the next result which, in particular, solves the problem mentioned in the introduction.

Theorem 3.8. *A longest alternating cycle in a multigraph $G \in \mathscr{E}\mathscr{C}\mathscr{M}$ with n vertices can be constructed in time $O(n^4)$.*

4. Proof of Theorem 3.5

To prove Theorem 3.5, we shall use the following lemma from [12,13]. We have stated the lemma in a more general way here, but it is easy to see from the proof in [12,13], that the proof actually covers this case also. Recall that a digraph D is called *strong* if, for every pair x, y of distinct vertices in D, there is a directed path from x to y and a directed path from y to x in D.

Lemma 4.1 (Gutin [12,13]). *Let D be a bipartite digraph obtained by taking two disjoint even directed cycles $C = u_1u_2\dots u_{2k-1}u_{2k}u_1$ and $Z = v_1v_2\dots v_{2r-1}v_{2r}v_1$ and adding an arc between v_{2i-1} and u_{2j} and between v_{2i} and u_{2j-1} (in any direction, possibly one in each direction) for all $i = 1,2,\dots,k$ and $j = 1,2,\dots,r$. D is Hamiltonian if and only if it is strong. Moreover, if D is strong, then, given cycles C and Z as above, a Hamiltonian directed cycle of D can be found in time $O(|V(C)||V(Z)|)$.*

In the statements of this section as well as in their proofs, we use the following notation: G is an extended 2-edge-coloured complete multigraph with n vertices, $\mathscr{F}_p = C_1 \cup \dots \cup C_p$ is a cycle subgraph in G consisting of p cycles, $C_1, \dots, C_p$; for each $i = 1,2,\dots,p$, $C_i = v_1^iv_2^i\dots v_{2k(i)}^iv_1^i$ such that $\chi(v_1^iv_2^i) = 1$, $\chi(v_{2k(i)}^iv_1^i) = 2$, and $X_i = \{v_1^i, v_3^i, \dots, v_{2k(i)-1}^i\}$, $Y_i = V(C_i) - X_i$. We write $C_j \to C_i$ to denote that $\chi(X_iX_i) = \chi(X_iV(C_j)), \chi(Y_iY_i) = \chi(Y_iV(C_j))$ and $\chi(X_iX_i) \neq \chi(Y_iY_i)$. We point out that the meaning of $C_j \to C_i$ is that, for any choice of vertices $x \in V(C_j)$ and $y \in V(C_i)$, there exist alternating (x, y)-paths P and P' such that the colours of the edges incident with x in P and P' are distinct, but for every such choice of paths P and P', the colours of the edges in P and P' incident with y are equal. Hence, if $C_j \to C_i$, then the multigraph induced by the vertices of these two cycles is not colour-connected.

In the special case when G is a 2-edge-coloured complete graph, the following two lemmas can be deduced from the results in [2,4].

Lemma 4.2. *Suppose G has a spanning cycle subgraph $\mathscr{F}_2 = C_1 \cup C_2$. Then, G is Hamiltonian if and only if neither $C_1 \rightarrow C_2$ nor $C_2 \rightarrow C_1$. Given a pair C_1 and C_2 of cycles of G, so that neither $C_1 \rightarrow C_2$ nor $C_2 \rightarrow C_1$, a Hamiltonian alternating cycle of G can be found in time $O(|V(C_1)||V(C_2)|)$.*

Proof. It is easy to see that if either $C_1 \rightarrow C_2$ or $C_2 \rightarrow C_1$, then G is not colour-connected. Hence, G is not Hamiltonian.

Assume that neither $C_1 \rightarrow C_2$ nor $C_2 \rightarrow C_1$, but G is not Hamiltonian.

First, observe that there is a complete connection between the vertices of C_1 and C_2, because if there is a pair of non-adjacent vertices, say v_i^1 and v_j^2, then one of the following two Hamiltonian cycles is alternating: $v_i^1 v_{j+1}^2 v_{j+2}^2 \dots v_j^2 P$, where $P = v_{i+1}^1 v_{i+2}^1 \dots v_i^1$ or $v_{i-1}^1 v_{i-2}^1 \dots v_i^1$.

Consider the bipartite digraph T with partite sets $V_1 = X_1 \cup X_2$ and $V_2 = Y_1 \cup Y_2$ obtained from G in the following way: delete all edges between vertices both on C_1 or on C_2 except those edges that are on the cycles and delete all edges between vertices both in the same partite set. Now, make the following orientations of the edges in the resulting bipartite multigraph. For $i = 1,2$ and any pair $v_1 \in V_1$, $v_2 \in V_2$, if there is an edge e between v_1 and v_2, then delete the colour of the edge e and orient it as the arc (v_i, v_{3-i}) iff $\chi(e) = i$. Obviously, T has a spanning cycle subgraph consisting of two directed cycles Z_1, Z_2 which are orientations of the cycles C_1, C_2, respectively. Similarly, we see that every directed cycle in T corresponds to an alternating cycle in G. Thus, since G is not Hamiltonian, T is not Hamiltonian either. By Lemma 4.1, this means that T is not strong, i.e. all arcs between Z_1 and Z_2 have the same orientation. W.l.o.g, we may assume that all these arcs are oriented from Z_1 to Z_2. Then, by the definition of T, we obtain that $\chi(X_1 Y_2) = 1$, $\chi(Y_1 X_2) = 2$.

Consider next the bipartite digraph T' with partite sets $V_1' = X_1 \cup Y_2$ and $V_2' = Y_1 \cup X_2$. The rest of the definition of T' coincides with that of T. T' also contains a spanning cycle subgraph consisting of orientations of C_1 and C_2. Since G is not Hamiltonian, T' is not Hamiltonian either. By Lemma 4.1, this means that T' is not strongly connected. This leads us to the conclusion that either $\chi(X_1 X_2) = 1$ and $\chi(Y_1 Y_2) = 2$ or $\chi(X_1 X_2) = 2$ and $\chi(Y_1 Y_2) = 1$. The first possibility together with the conclusion of the previous paragraph implies $\chi(X_1 V(C_2)) = 1$, $\chi(Y_1 V(C_2)) = 2$. The second gives $\chi(X_2 V(C_1)) = 2, \chi(Y_2 V(C_1)) = 1$. W.l.o.g, we may assume that $\chi(X_1 V(C_2)) = 1$, $\chi(Y_1 V(C_2)) = 2$.

Suppose that, for some $i \neq j$, there exists an edge $v_{2i+1}^1 v_{2j+1}^1$ of colour 2. Then G has the Hamiltonian alternating cycle

$$v_1^2 v_{2j}^1 v_{2j-1}^1 \dots v_{2i+1}^1 v_{2j+1}^1 \dots v_{2k(1)}^1 v_1^1 \dots v_{2i}^1 v_{2k(2)}^2 \dots v_1^2.$$

Hence, $\chi(X_1 X_1) = 1$. Analogously, $\chi(Y_1 Y_1) = 2$. Now, $C_2 \rightarrow C_1$ and we have obtained a contradiction.

The complexity bound follows from that of Lemma 4.1. □

A cycle subgraph $\mathscr{R}$ of G is called *irreducible* if there is no other cycle subgraph $\mathscr{Q}$ in G so that $V(\mathscr{R}) = V(\mathscr{Q})$ and $\mathscr{Q}$ has fewer cycles than $\mathscr{R}$.

Theorem 4.3. *Let G have a spanning cycle subgraph $\mathscr{F}$ consisting of $p \geqslant 2$ cycles. $\mathscr{F}$ is an irreducible spanning cycle subgraph of G if and only if we can label the cycles in $\mathscr{F}$ as $C_1, \ldots, C_p$, such that, with the notation introduced above, for every $1 \leqslant i < j \leqslant p$, $\chi(X_j V(C_i)) = 1$, $\chi(Y_j V(C_i)) = 2, \chi(X_j X_j) = 1, \chi(Y_j Y_j) = 2$. An irreducible spanning cycle subgraph of G (if any) can be found in time $O(n^{2.5})$.*

Proof. If the edges have the structure described above, then each of the cycles in $\mathscr{F}$ form a colour-connected component and $\mathscr{F}$ is clearly irreducible. To prove the other direction we let $\mathscr{F}$ be a irreducible spanning cycle subgraph of G and let $p \geqslant 2$ be the number of cycles in $\mathscr{F}$.

By Lemma 4.2, no two cycles in $\mathscr{F}$ induce a colour-connected subgraph. Thus, for all $1 \leqslant i < j \leqslant p$, either $C_i \rightarrow C_j$ or $C_j \rightarrow C_i$. Therefore, the digraph with vertex set $\{C_1, \ldots, C_p\}$ and arc set $\{(C_i, C_j)\colon C_i \rightarrow C_j; 1 \leqslant i \neq j \leqslant p\}$ is a tournament. So, if there exist cycles $C'_1, C'_2, \ldots, C'_k$ from $\mathscr{F}$ such that $C'_1 \rightarrow C'_2 \rightarrow \cdots \rightarrow C'_k \rightarrow C'_1$, then there also exists such a collection for $k = 3$ and the reader can easily find an alternating cycle covering precisely the vertices of those cycles, contradicting the irreducibility of $\mathscr{F}$. Hence, we can assume that there is no such cycle. Thus, there is a unique way to label the cycles in $\mathscr{F}$ as $C_1, C_2, \ldots, C_p$, so that $C_i \rightarrow C_j$ if and only if $i < j$. If there are three cycles C_i, C_j and C_k from $\mathscr{F}$ such that $C_i \rightarrow C_j, C_k$ and $C_j \rightarrow C_k$, but $\chi(X_k V(C_i)) \neq \chi(X_k V(C_j))$, then we can easily find an alternating cycle covering precisely the vertices of C_i, C_j and C_k, contradicting the irreducibility of $\mathscr{F}$. Hence, we may assume that for all $1 \leqslant i < j \leqslant p$, $\chi(X_j V(C_i)) = 1$ and $\chi(Y_j V(C_i)) = 2$. The fact that $\chi(X_j X_j) = 1, \chi(Y_j Y_j) = 2$ follows from the proof of Lemma 4.1.

Using the proof of Lemma 4.2, the proof above can be converted into an $O(n^2)$-algorithm similar to that in [14, p. 12.], Now, the complexity bound of the lemma follows from a simple fact that one can find a spanning cycle subgraph (if any) in a 2-edge-coloured multigraph L in time $O(|V(L)|^{2.5})$. Indeed, find maximum matchings in the red and blue subgraphs of L. Obviously, L has a spanning cycle subgraph iff both subgraphs have perfect matchings. The complexity bound follows from that of the algorithm for finding a maximum matching in a general graph described in [10]. □

Now, we are ready to give a proof of Theorem 3.5.

We will make use of the following simple lemma whose easy proof is left to the reader. We only point out that, under the conditions of Lemma 4.4, x_1 is adjacent to each vertex of C since $\chi(x_1 V(C)) = i$.

Lemma 4.4. *Let $P = x_1 x_2 \ldots x_k$ be an alternating path and C an alternating cycle disjoint from P in G. Suppose $\chi(x_1 V(C)) = i \neq \chi(x_1 x_2)$ where $i = 1$ or $i = 2$ and that G contains an edge $x_k z$, where $z \in V(C)$ and $\chi(x_{k-1} x_k) \neq \chi(x_k z)$. If $\chi(x_k z) = i$,* *then G contains a cycle C' with $V(C') = V(P) \cup V(C)$. Otherwise G has a cycle C'' with $V(C'') = V(P) \cup V(C) - w$, where w is the neighbour of z on C for which $\chi(wz) = 3 - i$.*

The following proof is shorter and simpler than the original proof of Corollary 3.6 in [18] since we show that there is always a 'suitable' alternating path from C_p to the rest of cycles in $\mathscr{F}$ (see the proof below).

Proof of Theorem 3.5. Let $\mathscr{F} = C_1 \cup \cdots \cup C_p$ be a cycle subgraph of G and let $\mathscr{F}' = C_1 \cup \cdots \cup C_{p-1}$. If $p = 1$, we are done. So, assume that $p \geqslant 2$. We shall show by induction on p that G has a cycle C^* covering at least the same number of vertices as $\mathscr{F}$. By Theorem 4.3, we may assume, using the (obvious) induction hypothesis, that, for all $1 \leqslant i < j \leqslant p$,

$$\chi(X_j V(C_i)) = 1, \qquad \chi(Y_j V(C_i)) = 2, \qquad \chi(X_j X_j) = 1, \qquad \chi(Y_j Y_j) = 2. \tag{1}$$

Note that, in particular, this implies that there is no pair of cycles $C', C'' \in \mathscr{F}$ so that there is a pair of nonadjacent vertices $x \in C'$ and $y \in C''$.

Since G is colour-connected there is an alternating (x,y)-path R of minimum length such that $x \in V(C_p)$, $\{y\} = V(R) \cap V(\mathscr{F}')$ and $\chi(xx') \neq \chi(xV(\mathscr{F}'))$, where x' is the successor of x in R. We prove that $(V(R) - \{x,y\}) \cap V(\mathscr{F}) = \emptyset$. Assume this is not so, that is R contains at least two vertices from C_p. Consider a vertex z in $(V(R) \cap V(C_p)) - x$. Let z' be the successor of z in R. Clearly, $\chi(zz') = \chi(zV(\mathscr{F}'))$ since the (z,y)-part of R is shorter than R. On the other hand, by (1) x' is not in C_p and by the minimality of R, $\chi(x'V(\mathscr{F}')) = \chi(xx')$. Then, the alternating path Qv, where Q is the reverse of the (x',z)-part of R and v is a vertex in C_{p-1}, is shorter than R; a contradiction.

Now, consider an alternating (x,y)-path R with the properties above including $(V(R) - \{x,y\}) \cap V(\mathscr{F}) = \emptyset$. We may assume w.l.o.g. that $x = v_1^p$ and $\chi(xV(\mathscr{F}')) = \chi(v_2^p v_1^p)$. Let $y \in V(C_t)$. Apply Lemma 4.4 to the path $v_{2k(p)}^p v_{2k(p)-1}^p \dots v_2^p R'$, where R' is the path R without y, and the cycle C_t. We get a new cycle C', with $V(C') \subset V(R) \cup V(C_t) \cup V(C_p)$, covering at least as many vertices as C_t and C_p together, so by replacing C_t and C_p by C' in $\mathscr{F}$, we obtain a new cycle subgraph with fewer cycles which cover at least as many vertices as $\mathscr{F}$ and the existence of C^* follows by induction.

The proof above can be converted into an $O(n^3)$-algorithm for finding a longest cycle in G, provided we are given a maximum cycle subgraph as input. The complexity bound of the theorem follows from this bound along with that of Theorem 3.7. □

5. Proofs of Theorems 3.3, 3.4 and 3.7

Proof of Theorem 3.7. Suppose that G is a 2-edge-coloured multigraph with n vertices. Form a weighted graph $H = H(G)$ as follows: $V(H) = \{x^1, x^2 : x \in V\}$ and, for every

$x \in V$, x^1x^2 is an edge of H and its weight is 2. Moreover, x^iy^i is an edge of H of weight 1 iff G has an edge of colour i with endvertices x and y. Let F be a minimum weight perfect matching of H and let F_1 be the set of all edges of F of weight 1. It is easy to see that F_1 corresponds to a maximum cycle subgraph T of G. Since a minimum weight perfect matching in a weighted graph on n vertices can be found in time $O(n^3)$ (cf. [17, Ch. 11]), we can construct T in time $O(n^3)$. □

It is easy to see that Theorem 3.4 follows from Proposition 5.1.

For a graph G and its matching M, a path P in G is *augmenting with respect to* M if, for any pair of adjacent edges in P, exactly one of them belongs to M, and the first and last edges of P do not belong to M.

Proposition 5.1. *Let $G = (V,E)$ be a connected 2-edge-coloured multigraph and let x and y be distinct vertices of G. For each choice of $i,j \in \{1,2\}$ we can find an alternating path $P = x_1x_2\dots x_k$ with $x_1 = x$, $x_k = y$, $\chi(x_1x_2) = i$ and $\chi(x_{k-1}x_k) = j$ in time $O(|E|)$ (if one exists).*

Proof. Let $W = V - \{x,y\}$ and create an uncoloured graph $G_{xy,ij}$ in the following way: $V(G_{xy,ij}) = \{x,y\} \cup W^1 \cup W^2$, where $W^r = \{z^r | z \in W\}$ for $r = 1,2$, $E(G_{xy,ij}) = \{xz^i | z \in W$ and $\chi(xz) = i\} \cup \{z^jy | z \in W$ and $\chi(zy) = j\} \cup \{u^kv^k | u,v \in W$ and $\chi(uv) = k\} \cup \{z^1z^2 | z \in W\}$.

The reader can easily verify that G has the desired path if and only if there exists an augmenting path in G_{xy} with respect to the matching $M = \{z^1z^2 | z \in W\}$. The latter can be checked, and a path constructed if one exists, in time $O(|E|)$ [19, p. 122]. From any augmenting path P in G_{xy} we can obtain the desired path in G, simply by contracting those edges of M which are on P. □

The rest of this section is a proof of Theorem 3.3.

Saad [18] proved that 2-edge-coloured complete graphs are convenient. Below we expand this result to extended 2-edge-coloured complete multigraphs. It is worth noting that our proof can be adopted to provide a considerably shorter proof of Saad's result above.

Let $\mathscr{P} = \{H_1,\dots,H_p\}$ be a set of subgraphs of a multigraph G. The *intersection* graph, $\Omega(\mathscr{P})$, of $\mathscr{P}$ has the vertex set $\mathscr{P}$ and the edge set $\{H_iH_j\colon V(H_i) \cap V(H_j) \neq \emptyset,\ 1 \leqslant i < j \leqslant p\}$. A pair, x,y, of vertices in a 2-edge-coloured multigraph H is called *cyclic connected* if H has a collection of alternating cycles $\mathscr{P} = \{C_1,\dots,C_p\}$ such that x and y belong to some cycles in $\mathscr{P}$ and $\Omega(\mathscr{P})$ is a connected graph.

The following lemma can be easily proved using only the definition of colour-connectivity. Lemma 5.2 provides a slightly simpler way of checking colour-connectivity.

Lemma 5.2. *A pair of vertices, x_1,x_2, in a 2-edge-coloured multigraph G is colour-connected if and only if G has four (x_1,x_2)-paths, $P_i^j = x_1v_i^j\dots u_i^jx_2$ ($i = 1,2$; $j = 1,2$), such that $\chi(x_1v_1^j) = \chi(u_2^jx_2) = j$ for $j = 1,2$.*

The following lemma shows that cyclic connectivity implies colour connectivity, even for general multigraphs.

Lemma 5.3. *If a pair, x, y, of vertices in a 2-edge-coloured multigraph G is cyclic connected, then x and y are colour-connected.*

Proof. If x and y belong to a common alternating cycle, then they are colour-connected. So, suppose that this is not the case.

Since x and y are cyclic connected, there is a collection $\mathscr{P} = \{C_1, \ldots, C_p\}$ of alternating cycles in G so that $x \in V(C_1)$, $y \in V(C_p)$, and, for every $i = 1, 2, \ldots, p-1$ and every $j = 1, 2, \ldots, p$, $|i-j| > 1$, $V(C_i) \cap V(C_{i+1}) \neq \emptyset$, $V(C_i) \cap V(C_j) = \emptyset$. ($\mathscr{P}$ corresponds to a shortest (C_1, C_p)-path in $\Omega(\mathscr{R})$, where $\mathscr{R}$ is the set of all alternating cycles in G.) We traverse $\mathscr{P}$ as follows: We start at the red (blue, resp.) edge of C_1 incident with x (from x) and go along C_1 to the first vertex u that belongs to both C_1 and C_2. After meeting u, we go along C_2 such that the path that we are forming will stay alternating. We repeat the procedure above when we meet the first vertex that belongs to both C_2 and C_3 and so on. Clearly, we shall eventually reach y. It follows that there is an (x, y)-path that starts from a red (blue, resp.) edge. By symmetry, we can construct an (x, y)-path that ends at a red (blue, resp.) edge. Hence, x and y are colour-connected by Lemma 5.2. $\square$

We formulate the following trivial but useful observation as a lemma. This observation shows that the notion of cyclic connectivity, in general, has some better properties than colour connectivity. However, we use colour connectivity in our treatment of Problem 3.2, since we do not know how to check whether a 2-edge-coloured multigraph is cyclic connected in polynomial time.

Lemma 5.4. *Cyclic connectivity is an equivalence relation on the vertices of a 2-edge-coloured multigraph.*

In the rest of this section, H denotes an extended 2-edge-coloured complete multigraph.

Lemma 5.5. *Let x and y be vertices in H and let $i, j \in \{1, 2\}$. If every alternating (x, y)-path that starts at an edge of colour i and ends at an edge of colour j has at least five vertices, then x and y are cyclic connected.*

Proof. Let $P = x_1 x_2 \ldots x_k$ be a shortest alternating (x, y)-path ($x_1 = x, x_k = y$) so that $\chi(x_1 x_2) = i$, $\chi(x_{k-1} x_k) = j$, $k \geqslant 5$. W.l.o.g. we may assume that $i = 1$.

Case 1: $i = j = 1$. If x and y are adjacent, then $\chi(xy) = 2$ and x, y belong to a common alternating cycle. Hence, we may assume that x and y are from the same partite set. If there is an edge e_1 between x_1 and x_4 in H, then $\chi(e_1) = 2$ since P is shortest. Analogously, if there is an edge e_2 between x_3 and x_k in H, then $\chi(e_2) = 2$.

Hence, if both e_1 and e_2 are in H, $x_1x_2x_3x_4x_1$ and $x_3x_4\dots x_kx_3$ are alternating cycles, i.e. x and y are cyclic connected. W.l.o.g. we may assume that e_1 is not in H. Since x and y are in the same partite set of H, $e_2 \in A(H)$. Then, $x_3x_4\dots x_kx_3$ is an alternating cycle. Since x_1 and x_4 belong to the same partite set, we can replace x_4 in the last cycle by x_1. Therefore, x, y belong to a common alternating cycle.

Case 2: $i = 1$, $j = 2$. If there is an edge e_1 between x_1 and x_{k-1} in H, then $\chi(e_1) = 2$ since P is shortest. By the same reasoning, if there is an edge e_2 between x_2 and x_k in H, then $\chi(e_2) = 1$. Hence, if H contains both e_1 and e_2, then x and y belong to two intersecting alternating cycles. Suppose that only one of e_1 and e_2, say e_1, is in H. Then x_2 and x_k are in the same partite set of H. Therefore, $x_1x_kx_3x_4\dots x_{k-1}x_1$ is an alternating cycle. If H contains none of the edges e_1 and e_2, then x_1 and x_{k-1} (x_2 and x_k, resp.) belong to the same partite set of H. This implies the existence of the following three alternating cycles: $x_1x_2x_1$, $x_{k-1}x_kx_{k-1}$ and $x_2x_{k-1}x_2$. Hence, x and y are cyclic connected. □

Lemma 5.6. *Let x and y be vertices in H. If x and y are colour-connected, then they are cyclic connected.*

Proof. Since x and y are colour-connected, there is a pair of alternating (x,y)-paths P and Q so that P (resp. Q) starts at an edge of colour i ($3-i$, resp.) and ends at an edge of colour j ($3-j$, resp.) We may assume that $|V(P)| \geqslant |V(Q)|$ and $|V(P)|$ is minimum possible. If $|V(P)| \geqslant 5$, we are done by Lemma 5.5. Suppose that $|V(P)| \leqslant 4$ and $i = j$. If x and y are not in the same alternating cycle, then x and y are in the same partite set and $|V(Q)| = 4$. A simple analysis of the remaining case shows that x and y are cyclic connected. If $|V(P)| \leqslant 4$ and $i \neq j$, then $3 \geqslant |V(P)| \geqslant |V(Q)|$. Clearly, x and y are cyclic connected. □

Lemmas 5.3, 5.4 and 5.6 imply immediately Theorem 3.3.

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